

# Entanglement negativity, Holography and Black holes

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## Abstract

Our recent conjecture for the holographic entanglement negativity is applied to the case of  $d$ -dimensional conformal field theories at finite temperatures dual to bulk  $AdS_{d+1}$ -Schwarzschild black holes in a generic  $AdS_{d+1}/CFT_d$  scenario. It is observed that the holographic entanglement negativity is related to the holographic mutual information and precisely captures the distillable quantum entanglement for the  $d$ -dimensional conformal field theories at finite temperatures. This non trivial example provides extremely strong evidence in favour of the universality of our proposed holographic conjecture.

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# 1 Introduction

The last decade has witnessed remarkable progress in the understanding of quantum entanglement in quantum information theory and has found application in diverse areas of theoretical physics and other related disciplines. This has inspired interesting insights in a plethora of disparate issues from quantum phase transitions to the black hole information loss paradox. For a bipartite quantum system  $(A \cup B)$  with a Hilbert space  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ , the pure state entanglement may be understood in terms of the existence of Bell states or Einstein-Podolsky-Rosen (EPR) pairs. A measure of this entanglement is provided for a bipartite system in a pure state  $|\psi_{AB}\rangle$ , through the quantity known as the *entanglement entropy*. The entanglement entropy may be measured by the von-Neumann entropy of the reduced density matrix  $\rho_A = \text{Tr}_B(\rho_{A \cup B})$  of the subsystem  $A$ . For few particle quantum systems this quantity may be computed in a straightforward manner. On the contrary the issue of characterizing entanglement for extended quantum many body systems in arbitrary dimensions is extremely complex and has seen very little progress. However, for  $(1+1)$ -dimensional quantum field theory with conformal symmetry this issue is rendered tractable. As demonstrated by Calabrese and Cardy [1, 2] in a seminal contribution, the entanglement entropy for such a  $(1+1)$ -dimensional conformal field theory (CFT) may be obtained through the *replica technique*. This technique is based on the simple idea of computing the moments of the reduced density matrix  $\text{Tr}(\rho_A^n)$  with  $n$  being a non-negative integer or equivalently the Rényi entropy of order  $n$  which may be defined as

$$S_A^{(n)} = \frac{\ln[\text{Tr}(\rho_A^n)]}{1-n}. \quad (1)$$

The quantity  $\text{Tr}(\rho_A^n)$  in this computation corresponds to the partition function of the subsystem  $A$  in the  $CFT_{1+1}$  on a  $n$ -sheeted Riemann surface with branch points at the boundaries between the subsystems  $A$  and  $B$  [1]. Note that the corresponding von-neumann entropy may be obtained from the above expression for the Rényi entropy through the replica limit  $n \rightarrow 1$  which reduces the  $n$ -sheeted Riemann surface to the complex plane. Furthermore, the partition function for the subsystem on the  $n$ -sheeted Riemann surface may be recast in terms of the correlation functions of branch-point twist fields on the complex plane [1, 2] in this limit. The corresponding correlation functions may then be computed in a straightforward fashion in the  $CFT_{1+1}$  to obtain the entanglement entropy.

Notice however that the entanglement entropy for a mixed state of a bipartite quantum system at a finite temperature receives contribution from both the classical (thermal) and the quantum correlations. This makes it unsuitable as a valid measure for obtaining the distillable quantum entanglement for a bipartite quantum system in a mixed state at a finite temperature. The issue of the removal of the thermal correlations in a process termed as *purification* is a central issue in quantum information theory which leads to the distillable quantum entanglement for a mixed state at a finite temperature. In a classic work Vidal and Werner [3] introduced a computable measure for such a mixed-state entanglement in bipartite quantum systems, termed as the *entanglement negativity*. This measure involves a partial transpose of the full density matrix over one of the subsystems in a bipartite quantum system. In order to define entanglement negativity it is required to consider an extended quantum system which is divided into two parts  $A_1$  and  $A_2$ <sup>1</sup>. If  $|q_i^1\rangle$  and  $|q_i^2\rangle$  represent the bases of Hilbert space corresponding to the subsystems  $A_1$  and  $A_2$  respectively, then the partial transpose with respect to  $A_2$  degrees of freedom is expressed as

$$\langle q_i^1 q_j^2 | \rho_{A_1 \cup A_2}^{T_2} | q_k^1 q_l^2 \rangle = \langle q_i^1 q_l^2 | \rho_{A_1 \cup A_2} | q_k^1 q_j^2 \rangle, \quad (2)$$

where,  $\rho_{A_1 \cup A_2}$  is the density matrix of the system ( $A = A_1 \cup A_2$ ). This leads to the definition of

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<sup>1</sup>In principle the computation of entanglement negativity is extremely subtle involving a procedure referred to as purification in quantum information theory. This requires obtaining a mixed state by tracing out the degrees of freedom of a larger system in a pure state. For instance if the full system is divided in to three parts say  $A_1, A_2$  and  $B$  then the required density matrix  $\rho_{A_1 \cup A_2}$  is obtained by tracing over the subsystem  $B$ .

the entanglement negativity as

$$\mathcal{E} \equiv \log (Tr | \rho_{A_1 \cup A_2}^{T_2} | ) = \log (Tr | \rho_A^{T_2} | ). \quad (3)$$

Observe that from the above equation, the entanglement negativity may be expressed as the logarithm over the sum of the absolute eigenvalues of the density matrix  $\rho_A^{T_2}$ . This may be written as follows

$$Tr | \rho_A^{T_2} | = \sum_{\lambda_i > 0} |\lambda_i| + \sum_{\lambda_i < 0} |\lambda_i|, \quad (4)$$

where  $\lambda_i$  correspond to the eigenvalues of the density matrix  $\rho_A^{T_2}$ . Remarkably, the entanglement negativity measures the distillable quantum entanglement for an extended quantum system in a mixed state. The entanglement negativity exhibits certain important properties including the significant properties of non-convexity and monotonicity which were proved by Plenio in [4]. Recently, the issue of obtaining the distillable quantum entanglement for  $(1+1)$ -dimensional conformal field theories at finite temperatures has received considerable attention (See [5] for references and reviews). In [6–8] the authors have advanced a systematic procedure to compute the entanglement negativity for  $(1+1)$ -dimensional CFTs at finite temperatures. Their procedure involves the *replica technique* mentioned earlier to compute the logarithm of a certain four point function of the twist and the anti-twist fields that leads to the entanglement negativity. Through this the authors were able to demonstrate that the *entanglement negativity* measures the distillable quantum entanglement at finite temperatures through the subtraction of the thermal contributions.

In a different context the *AdS/CFT* correspondence has established a holographic duality between a weakly coupled theory of gravity in a bulk *AdS* space time and a strongly coupled conformal field theory on the boundary [9–12]. In this framework, a bulk *AdS* black hole was shown to be dual to a boundary conformal field theory at a finite temperature. In [13, 14] Ryu and Takayanagi conjectured a holographic prescription which may be used to obtain the quantum entanglement entropy for boundary conformal field theories in arbitrary dimensions. This prescription for obtaining the entanglement entropy  $S_A$  for a region  $A$  (enclosed by the boundary  $\partial A$ ) in the  $(d)$ -dimensional boundary conformal field theory involves the computation of the area of the extremal surface (denoted by  $\gamma_A$ ) extending from the boundary  $\partial A$  of the region  $A$  into the  $(d+1)$ -dimensional bulk which may be expressed as

$$S_A = \frac{Area(\gamma_A)}{(4G_N^{(d+1)})}, \quad (5)$$

where,  $G_N^{(d+1)}$  is the gravitational constant of the bulk space time. Application of this holographic prescription to compute the entanglement entropy for various strongly coupled boundary CFTs at both zero and finite temperatures has yielded interesting insights [5, 15–19].

The crucial issue which emerges from the above discussion is that of a corresponding holographic prescription for the entanglement negativity of such boundary conformal field theories at finite temperatures in the context of the *AdS/CFT* correspondence. This critical issue has received certain attention recently and has led to interesting insights. In this context, in [20] the authors have computed the entanglement negativity of zero temperature conformal field theories dual to the bulk vacuum *AdS* spacetime in arbitrary dimensions. Furthermore in [21] the authors have conjectured a holographic *c*-function which involves the causal horizon of the bulk black brane geometry that is identified with the entanglement negativity of the dual conformal field theory at a finite temperature. Very recently we; Chaturvedi, Malvimat and Sengupta (CMS) have conjectured an elegant holographic prescription for the entanglement negativity of boundary conformal field theories in arbitrary dimensions at finite temperatures [22]. Interestingly, the entanglement negativity computed from the bulk is directly related to the *holographic mutual*

information in our conjecture. In the context of the  $AdS_3/CFT_2$  scenario involving a bulk Euclidean BTZ black hole, our holographic prescription exactly reproduces the large central charge  $c$  limit for the entanglement negativity of the  $CFT_{1+1}$  described in [8]. This clearly validates our conjecture in this case.

In this article we employ the conjecture proposed by CMS [22] to compute the entanglement negativity in the context of  $AdS_{d+1}/CFT_d$  which involves a bulk geometry describing a  $(d+1)$ -dimensional  $AdS$ -Schwarzschild black hole. We will show that the holographic entanglement negativity thus obtained once again captures the distillable quantum entanglement for the  $d$  dimensional boundary conformal field theory at a finite temperature. This constitutes extremely strong evidence for the universality of our holographic conjecture for the entanglement negativity of finite temperature conformal field theories in arbitrary dimensions in the  $AdS/CFT$  framework. We would like to emphasize that this is an extremely significant development which is expected to have wide ramifications in several diverse areas including quantum phase transitions, quantum information theory and the information loss paradox for black holes amongst others.

In Section two, we briefly review the results of [8] for the entanglement negativity of  $(1+1)$  dimensional conformal field theories at finite temperatures. In Section three we discuss the large- $c$  limit of the entanglement negativity and its relevance in establishing our holographic conjecture. Subsequently in Section four, we describe our conjecture for the holographic entanglement negativity for CFTs in arbitrary dimensions and briefly review its application in the  $AdS_3/CFT_2$  scenario [22]. In Section five, we employ our holographic conjecture to obtain the entanglement negativity of  $d$ -dimensional boundary CFTs at finite temperatures which are dual to  $AdS_{d+1}$ -Schwarzschild black holes and demonstrate that this captures the distillable quantum entanglement through the removal of the thermal contribution. In the last section we provide a summary of our results and discuss future applications of our conjecture and its significance in diverse areas.

## 2 Entanglement entropy and entanglement negativity in $CFT_{1+1}$

In this section we begin by briefly reviewing the procedure for obtaining the entanglement entropy of  $(1+1)$  dimensional CFTs at finite temperatures. Subsequently we elucidate the issue of the inadequacy of the entanglement entropy as a valid measure for the distillable quantum entanglement at finite temperatures. Furthermore we review the introduction of the *entanglement negativity* as a valid measure for the distillable quantum entanglement of  $(1+1)$ -dimensional CFTs at finite temperatures. For completion, we have reviewed the technique for obtaining the entanglement negativity in  $CFT_{1+1}$  and its relation to the four point function of the twist and anti-twist operators in the Appendix.

### 2.1 Entanglement entropy

In general entanglement entropy is considered as a measure for quantum entanglement in a bipartite quantum system. Considering the subsystem  $A$  and its complement  $A^c$  of such an extended bipartite quantum system the entanglement entropy for the subsystem  $A$  is given as

$$S_A = \lim_{n \rightarrow 1} \frac{\ln(\text{Tr}[\rho_A^n])}{1-n} = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \text{Tr}[\rho_A^n], \quad (6)$$

where,  $\rho$  is the full density matrix and  $\rho_A = \text{Tr}_{A^c}(\rho)$  denotes the reduced density matrix for the subsystem- $A$  and  $n \rightarrow 1$  is the replica limit. For a  $(1+1)$ -dimensional CFT the moments of the reduced density matrix  $\text{Tr}(\rho_A^n)$  are related to the partition function on a  $n$ -sheeted Riemann surface with branch points at the boundaries between regions  $A$  and  $A^c$  [1]. Alternatively, the partition function on a  $n$ -sheeted Riemann surface may be recast as the two point function of the branch-point twist/anti-twist fields  $\mathcal{T}_n$  and  $\bar{\mathcal{T}}_n$  on the complex plane with the following scaling

dimensions

$$\Delta_n = \frac{c}{12}(n - 1/n), \quad (7)$$

here,  $c$  is the central charge of the CFT. Hence following [1, 2] the general form for the quantity  $Tr\rho_A^n$  may be expressed as follows

$$Tr\rho_A^n = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1) \cdots \mathcal{T}_n(u_N)\bar{\mathcal{T}}_n(v_N) \rangle, \quad (8)$$

where,  $A = \cup_{i=1}^N [u_i, v_i]$  indicates that the subsystem  $A$  has been divided into  $N$  disjoint intervals. For the case when  $N = 1$  with the subsystem length  $|u - v| = \ell$ , the eq.(8) reduces to the following

$$Tr\rho_A^n = \langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v) \rangle = c_n \left( \frac{\ell}{a} \right)^{-c/6(n-1/n)}, \quad (9)$$

here,  $c_n$  is some constant and  $a$  is the  $UV$  cut-off for the  $(1 + 1)$ -dimensional CFT. Using the expression for the entanglement entropy in eq.(6) along with the eq.(9) leads to the following result

$$S_A = \frac{c}{3} \ln \left( \frac{\ell}{a} \right) + constant. \quad (10)$$

The above result corresponds to the zero temperature entanglement entropy of a subsystem  $A$  with length  $\ell$  in the  $(1 + 1)$ - dimensional CFT. For the corresponding result at finite temperatures it is required to evaluate the two point function in eq. (9) on a cylinder of circumference  $\beta = 1/T$  such the cylinder is sewed everywhere except for a branch cut along the interval representing subsystem  $A$  [1, 2]. The above procedure leads to the following expression for the finite temperature entanglement entropy of subsystem  $A$  as

$$S_A = \frac{c}{3} \log \left( \frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta} \right) + constant. \quad (11)$$

From the  $AdS/CFT$  correspondence the results for the entanglement entropy given by eq.(10) and eq.(11) may be obtained by using the Ryu and Takyanagi conjecture [13, 14, 16]. Note that the Ryu and Takyanagi conjecture provides a systematic procedure to obtain the entanglement entropy for higher dimensional boundary CFTs at finite temperature and charge density which are dual to bulk  $AdS$ - Schwarzschild/Reissner-Nordstrom black holes (See [17, 19, 23–25] and references therein).

Observe that from eq.(11) the large temperature limit leads to the purely thermal entropy indicating that the entanglement entropy receives contribution from both the classical ( thermal) and the quantum correlations at finite temperatures. A similar observation may also be made for the case of higher dimensional conformal field theories at finite temperatures dual to bulk  $AdS$  black holes in the context of the Ryu and Takayanagi conjecture [17, 19]. Notice that as mentioned earlier this is a generic issue for any quantum system in quantum information theory and renders the entanglement entropy unsuitable as an entanglement measure at finite temperatures. This naturally leads to the question of establishing appropriate measures to characterize the distillable quantum entanglement at finite temperatures. As indicated earlier the entanglement negativity introduced by Vidal and Werner [3] constitutes such an appropriate measure for this purpose which we now proceed to describe in the context of  $(1 + 1)$ -dimensional CFTs.

## 2.2 Entanglement negativity in $CFT_{(1+1)}$

In order to define entanglement negativity in  $(1 + 1)$ -dimensional CFTs it is required to consider the partition  $A_1, A_2$  and  $A^c$  such that  $A_1$  and  $A_2$  correspond to finite intervals  $[u_1, v_1]$  and  $[u_2, v_2]$  of lengths  $\ell_1$  and  $\ell_2$  respectively whereas,  $A^c$  represents the rest of the system. Let  $\rho_A$  denote the reduced density matrix of the subsystem  $A = A_1 \cup A_2$  such that  $\rho_A = \rho_{A_1 \cup A_2}$  which is obtained

by tracing out the full density matrix  $\rho$  over the part  $A^c$ , i.e.  $\rho_A = \text{Tr}_{A^c}(\rho)$ . As mentioned earlier in the Introduction the entanglement negativity is then given as

$$\mathcal{E} = \ln \left[ \text{Tr} \left| \rho_A^{T_2} \right| \right], \quad (12)$$

where  $\rho_A^{T_2}$  is as given in eq.(2).

The authors in [8] employed the replica technique to show that the entanglement negativity  $\mathcal{E}$  for  $(1+1)$ -dimensional CFTs may be expressed as follows

$$\mathcal{E} = \lim_{n_e \rightarrow 1} \ln \left[ \text{Tr}(\rho^{T_A})^{n_e} \right], \quad (13)$$

where, the trace norm is required to be evaluated as an analytic continuation of a sequence  $\{n_e\}$  where  $n_e$  is an even number to the replica limit  $n_e \rightarrow 1$  and  $\rho = \rho_{A \cup A^c}$  corresponds to the full density matrix. The mathematical details of the transition from a tripartite configuration  $(A_1, A_2, A^c)$  to a bipartite configuration  $(A, A^c, \emptyset)$  are reviewed in the Appendix.

It follows that the entanglement negativity for  $(1+1)$ -dimensional CFT at zero temperature is obtained through the replica limit of a certain two point function of the twist and anti-twist fields evaluated at the end points of the subsystem- $A$  under consideration i.e

$$\mathcal{E} = \lim_{n_e \rightarrow 1} \ln \left( \langle \mathcal{T}_{n_e/2}(u) \bar{\mathcal{T}}_{n_e/2}(v) \rangle \right)^2, \quad (14)$$

where  $\mathcal{T}_{n_e/2}$  and  $\bar{\mathcal{T}}_{n_e/2}$  correspond to the twist and the anti twist fields respectively with scaling dimensions  $(\Delta_{n_e/2})$  given as

$$\Delta_{n_e/2} = \frac{c}{12} \left( \frac{n_e}{2} - \frac{2}{n_e} \right). \quad (15)$$

Substituting the well known form for the two point function under consideration in eq.(14) one arrives at the following result

$$\mathcal{E} = \frac{c}{2} \ln \left( \frac{\ell}{a} \right) + \text{constant}. \quad (16)$$

The result matches with the expectation from quantum information theory that the entanglement negativity is the Rényi-entropy of order half at zero temperature. Furthermore, the authors also showed that at a finite temperature when the system is in a mixed state, the entanglement negativity is related to a certain four point function of the twist and the anti-twist fields as follows

$$\mathcal{E} = \lim_{L \rightarrow \infty} \lim_{n_e \rightarrow 1} \ln \left[ \langle \mathcal{T}_{n_e}(-L) \bar{\mathcal{T}}_{n_e}^2(-\ell) \mathcal{T}_{n_e}^2(0) \bar{\mathcal{T}}_{n_e}(L) \rangle_\beta \right], \quad (17)$$

where the subscript  $\beta$  indicates that the above four point function has to be computed for a finite temperature as described earlier on an infinite cylinder with circumference  $\beta$ . The scaling dimension  $(\Delta_{n_e}^{(2)})$  of the operator  $\mathcal{T}_{n_e}^2$  may be related to the scaling dimensions  $(\Delta_{n_e})$  of the operator  $\mathcal{T}_{n_e}$  as follows

$$\begin{aligned} \Delta_{n_e}^{(2)} = 2\Delta_{n_e/2} &= \frac{c}{6} \left( \frac{n_e}{2} - \frac{2}{n_e} \right), \\ \Delta_{n_e} &= \frac{c}{12} \left( n_e - \frac{1}{n_e} \right). \end{aligned} \quad (18)$$

Evaluating the four point function given in eq.(17) it could be shown that the finite temperature entanglement negativity in  $CFT_{1+1}$  may be expressed as

$$\mathcal{E} = \frac{c}{2} \ln \left[ \frac{\beta}{\pi a} \sinh \left( \frac{\pi \ell}{\beta} \right) \right] - \frac{\pi c \ell}{2\beta} + f(e^{-2\pi \ell / \beta}) + \ln(c_{1/2}^2 c_1). \quad (19)$$

Here  $c_{1/2}$  and  $c_1$  are constants whose values may be set to unity by choosing a proper normalization for the two-point function of the twist operators (See Appendix for details of the above computations). The function  $f(x)$  where  $x = e^{-2\pi\ell/\beta}$  is a non universal part which remains undetermined in a  $CFT_{1+1}$  and depends on the full operator content of the theory. For brevity the above eq.(19) may be re-expressed as

$$\mathcal{E} = \frac{3}{2} \left[ S_A - S_A^{th} \right] + f(e^{-2\pi\ell/\beta}), \quad (20)$$

where  $S_A = \frac{c}{3} \ln \left[ \frac{\beta}{\pi a} \sinh \left( \frac{\pi\ell}{\beta} \right) \right]$  corresponds to the entanglement entropy and  $S_A^{th} = \frac{\pi c \ell}{3\beta}$  to the thermal entropy of the subsystem- $A$ . This is an extremely significant result for  $CFT_{1+1}$  at a finite temperature which illustrates that the negativity  $\mathcal{E}$  captures the distillable quantum entanglement through the removal of the thermal contribution. In the next section, we will discuss the large central charge ( $c$ ) limit of the above result and its significance in the context of the  $AdS/CFT$  correspondence.

### 3 Large- $c$ limit of the entanglement negativity in $CFT_{1+1}$

In this section, we discuss the large- $c$  ( $c \rightarrow \infty$ ) limit for the corresponding four point function of the twist/anti-twist fields related to the entanglement negativity for a  $CFT_{1+1}$  at a finite temperature in eq.(20). We will demonstrate that in this limit the leading dominant contribution to the entanglement negativity arises from the universal term ( $S_A - S_A^{th}$ ) whereas the non universal term given by the unknown function  $f(x)$  in eq.(20) is sub leading.

The evidence for the above statement arises from the study of the conformal block expansion for the four point function in a  $(1+1)$ -dimensional CFT. Consider a four point function of primary operators  $\mathcal{O}_i$  ( $i = 1, 2, 3, 4$ ) such that each operator  $\mathcal{O}_i$  acts at a point  $z_i$  on the complex plane, and has the scaling dimension denoted by  $\Delta_i$ . In  $CFT_{1+1}$  it is always possible to make a conformal transformation  $w = \frac{(z-z_1)(z_3-z_4)}{(z-z_4)(z_3-z_1)}$  such that the points  $(z_1, z_2, z_3, z_4) \rightarrow (0, x, 1, \infty)$ . Note here that  $x$  is the cross ratio given by  $x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ . Under this transformation the four point function may be expanded in terms of the conformal blocks as follows

$$G(x) = \lim_{|w_4| \rightarrow \infty} |w_4|^{2\Delta_4} \langle \mathcal{O}_1(0) \mathcal{O}_2(x) \mathcal{O}_3(1) \mathcal{O}_4(w_4) \rangle = \sum_p a_p \Psi(h_i, h_p, x) \bar{\Psi}(\bar{h}_i, \bar{h}_p, \bar{x}). \quad (21)$$

Here,  $h_i$  and  $\bar{h}_i$  are the holomorphic and the anti-holomorphic scaling dimensions such that  $h_i + \bar{h}_i = \Delta_i$ . The summation in the above equation is over all the primary operators  $\mathcal{O}_p$  with scaling dimensions  $h_p$  and  $\bar{h}_p$  and  $\Psi(h_i, h_p, x)$ ,  $\bar{\Psi}(\bar{h}_i, \bar{h}_p, \bar{x})$  are the corresponding conformal blocks. In recent years there has been considerable efforts to elucidate the large- $c$  limit of this conformal block expansion for the four point function. There is strong evidence that the four point function of a  $CFT_{1+1}$  exponentiates in the large- $c$  limit and is universal [26]. This result is based on some earlier studies by Zamolodchikov in the context of the semi-classical (large  $c$ ) limit of the conformal blocks in Liouville field theory [27, 28]. Therefore, in the large- $c$  ( $c \rightarrow \infty$ ) limit with  $\frac{h_i}{c}$  and  $\frac{\bar{h}_p}{c}$  held fixed, the conformal block and hence the four point function exponentiates as follows

$$G(x) \approx \exp \left[ -\frac{c}{6} \left[ g\left(\frac{h_i}{c}, \frac{h_p}{c}, x\right) + \bar{g}\left(\frac{\bar{h}_i}{c}, \frac{\bar{h}_p}{c}, \bar{x}\right) \right] \right]. \quad (22)$$

Note that this is a semi-classical result valid in the large central charge limit ( $c \rightarrow \infty$ ) and there are  $O[\frac{1}{c}]$  corrections to this. This property has been used to study the large- $c$  limit of the entanglement entropy of two disjoint intervals in  $CFT_{1+1}$  which is also a certain four point function of the twist and the anti-twist fields [29–32]. In these articles the authors have shown that the leading contribution to the four point function arises from the universal terms in the large- $c$  limit whereas the non universal terms are sub leading (i.e  $O[c^0]$  or lesser). The



Ryu-Takayanagi conjecture for the case of two disjoint intervals precisely reproduces this leading term in the large- $c$  limit of the four point function in the boundary  $CFT_{1+1}$ . Observe that the above argument applies to any generic four point function in  $CFT_{1+1}$ , in particular the four point function of the twist and the anti-twist fields which defines the entanglement negativity in eq. (17). Hence this implies that in the large  $c$  limit the non-universal term given by the function  $f(x)$  in eq.(19) for the entanglement negativity is sub leading with the leading contribution arising from the universal part ( $S_A - S_A^{th}$ ).

Furthermore, in quantum information theory there is a well known phenomenon termed *de-coherence* which causes the entanglement to decay rapidly with the interaction between a system and its environment ( for our case this indicates an increase in the temperature through interaction with a heat reservoir). This implies that the entanglement negativity which is a measure for the distillable quantum entanglement must also decrease with the temperature and is bounded from above by its zero temperature value which results in the following inequality

$$\mathcal{E}_{T=0} > \mathcal{E}_{T \neq 0}. \quad (23)$$

Hence, using the expressions for the values of the entanglement negativity at zero and at finite temperature given by eq.(16) and eq.(19) respectively leads to an upper bound on the function  $f(x)$  which is given by

$$f(x) < \frac{\pi c \ell}{2\beta} - \frac{c}{2} \ln \left[ \frac{\beta}{\pi \ell} \sinh \left( \frac{\pi \ell}{\beta} \right) \right]. \quad (24)$$

Moreover the positivity property of the entanglement negativity introduces a lower bound on the function  $f(x)$  which may be stated as

$$f(x) \geq \frac{\pi c \ell}{2\beta} - \frac{c}{2} \ln \left[ \frac{\beta}{\pi a} \sinh \left( \frac{\pi \ell}{\beta} \right) \right]. \quad (25)$$

From eq.(24) and eq.(25) we conclude that the function  $f(x)$  is bounded from both below and above.

The above arguments from the perspective of the  $CFT_{1+1}$  and Quantum Information Theory indicates that the function  $f(x)$  describing the non universal part of the finite temperature entanglement negativity is both sub leading in the large central charge limit and bounded. Hence in this limit the entanglement negativity given by eq (19) reduces to the following expression

$$\mathcal{E} = \frac{c}{2} \ln \left[ \frac{\beta}{\pi a} \sinh \left( \frac{\pi \ell}{\beta} \right) \right] - \frac{\pi c \ell}{2\beta} = \frac{3}{2} \left[ S_A - S_A^{thermal} \right]. \quad (26)$$

The above equation clearly demonstrates that in the large central charge limit the entanglement negativity for a  $CFT_{1+1}$  assumes this universal form describing the removal of the thermal contribution and leading to the distillable quantum entanglement. In the context of the  $AdS/CFT$  correspondence the large central charge limit essentially describes the large  $N$  limit of the boundary CFT through the Brown-Henneaux formula [33, 34]. This leads us to the possibility of a corresponding holographic conjecture for the entanglement negativity in the  $AdS/CFT$  scenario. As mentioned earlier, the authors Chaturvedi, Malvimat and Sengupta ( CMS) in [22] proposed such a holographic conjecture which exactly reproduces the above result in eq. (26) from a bulk computation which involves a Euclidean BTZ black hole in the  $AdS_3/CFT_2$  scenario. This is briefly reviewed in the following section.

## 4 Holographic prescription for the entanglement negativity

In this section, we review the holographic prescription provided in [22] for the entanglement negativity of a  $(1+1)$ -dimensional finite temperature CFT in the  $AdS_3/CFT_2$  scenario. To begin with let us consider the boundary  $CFT_{1+1}$  to be partitioned into the subsystem  $A$  and its

complement  $A^c$ . We denote  $B_1$  and  $B_2$  as two large finite intervals adjacent to  $A$  on either side of it such that  $B = B_1 \cup B_2$  as shown in fig.(1). As mentioned in section-2, the entanglement negativity is defined in the limit  $B \rightarrow A^c$  ( $L \rightarrow \infty$ ) which corresponds to extending the subsystems  $B_1$  and  $B_2$  to infinity.

The two point function of the twist fields  $(\mathcal{T}_{n_e}, \bar{\mathcal{T}}_{n_e})$  in a  $(1+1)$ -dimensional boundary CFT at a finite temperature may be expressed as follows<sup>2</sup>

$$\langle \mathcal{T}_{n_e}(z_i) \bar{\mathcal{T}}_{n_e}(z_j) \rangle_{\mathbb{C}} = \frac{c_{n_e}}{z_{ij}^{\Delta_{n_e}}}, \quad (27)$$

where,  $z_{ij} = |z_i - z_j|$  and  $c_{n_e}$  is a constant. From the  $AdS/CFT$  dictionary the two point function on the boundary  $CFT_{1+1}$  may be related to the length of the geodesic  $\mathcal{L}_{ij}$  anchored on the points  $(z_i, z_j)$  and extending into the bulk  $AdS_{2+1}$  space time as follows

$$\langle \mathcal{T}_{n_e}(z_i) \bar{\mathcal{T}}_{n_e}(z_j) \rangle_{\mathbb{C}} \sim e^{-\frac{\Delta_{n_e} \mathcal{L}_{ij}}{R}}, \quad (28)$$

where,  $R$  is the  $AdS$  radius of the bulk  $AdS_{2+1}$  space time. From fig.(1) one may identify that

$$\mathcal{L}_{12} = \mathcal{L}_{B_1}, \quad \mathcal{L}_{23} = \mathcal{L}_A, \quad \mathcal{L}_{34} = \mathcal{L}_{B_2}, \quad \mathcal{L}_{13} = \mathcal{L}_{A \cup B_1}, \quad \mathcal{L}_{24} = \mathcal{L}_{A \cup B_2}, \quad \mathcal{L}_{14} = \mathcal{L}_{A \cup B}. \quad (29)$$

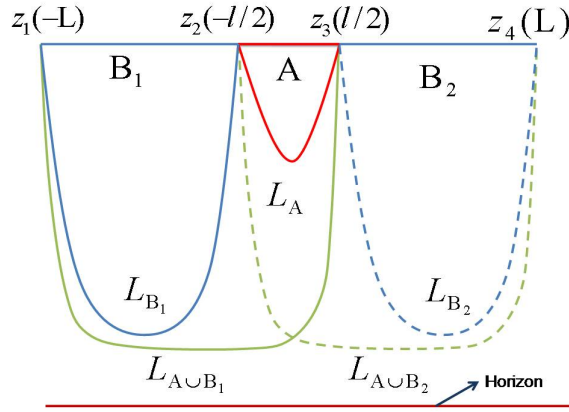


Figure 1: Schematic of geodesics anchored on the subsystems  $A$ ,  $B_1$  and  $B_2$  which live on the  $(1+1)$ -dimensional boundary.

Using eq.(28) and eq.(29), the four point function of the twist fields given by eq.(93) in the Appendix, reduces to the following form in terms of the geodesic lengths as

$$\langle \mathcal{T}_{n_e}(z_1) \bar{\mathcal{T}}_{n_e}^2(z_2) \mathcal{T}_{n_e}^2(z_3) \bar{\mathcal{T}}_{n_e}(z_4) \rangle_{\mathbb{C}} \sim e^{-\frac{\Delta_{n_e} \mathcal{L}_{A \cup B}}{R}} e^{-\frac{\Delta_{n_e}^{(2)} \mathcal{L}_A}{R}} \frac{\mathcal{F}_{n_e}(x)}{x^{\Delta_{n_e}^{(2)}}}. \quad (30)$$

Here the cross ratio  $x$  may also be defined in terms of the geodesic lengths as

$$x^{\Delta_{n_e}^{(2)}} = e^{\frac{\Delta_{n_e}^{(2)} (\mathcal{L}_{B_1} + \mathcal{L}_{B_2} - \mathcal{L}_{A \cup B_1} - \mathcal{L}_{A \cup B_2})}{2R}}. \quad (31)$$

In the replica limit  $n_e \rightarrow 1$ , we have  $\Delta_{n_e} \rightarrow 0$  and  $\Delta_{n_e}^{(2)} \rightarrow -\frac{c}{4}$  such that the four point function given by eq.(30) reduces to the following form

$$\lim_{n_e \rightarrow 1} \langle \mathcal{T}_{n_e}(z_1) \bar{\mathcal{T}}_{n_e}^2(z_2) \mathcal{T}_{n_e}^2(z_3) \bar{\mathcal{T}}_{n_e}(z_4) \rangle_{\mathbb{C}} \sim e^{\frac{c}{8R} (2\mathcal{L}_A + \mathcal{L}_{B_1} + \mathcal{L}_{B_2} - \mathcal{L}_{A \cup B_1} - \mathcal{L}_{A \cup B_2}) + f(x)}. \quad (32)$$

<sup>2</sup>As stated earlier for a the finite temperature case the two point function must be evaluated on an infinitely long cylinder of circumference  $\beta = 1/T$  which may be obtained through the conformal transformation,  $z \rightarrow \omega = \beta/2\pi \ln z$ .

It is to be noted that the central charge ‘ $c$ ’ of  $CFT_{1+1}$  is related to the AdS length  $R$  through the Brown-Henneaux formula  $c = \frac{3R}{2G_N^3}$ , where  $G_N^3$  is the  $(2+1)$ -dimensional gravitational constant [35]. Note that as discussed in the previous section the sub leading non-universal term given by the function  $f(x)$  may be neglected in the semi classical large- $c$  limit of the four point function. Therefore, using eq.(17) and the above mentioned Brown-Henneaux formula one may express the holographic entanglement negativity for the boundary  $CFT_{1+1}$  at a finite temperature as follows

$$\mathcal{E} = \lim_{B \rightarrow A^c} \left[ \frac{3}{16G_N^3} (2\mathcal{L}_A + \mathcal{L}_{B_1} + \mathcal{L}_{B_2} - \mathcal{L}_{A \cup B_1} - \mathcal{L}_{A \cup B_2}) \right]. \quad (33)$$

In the  $AdS_3/CFT_2$  scenario the Ryu and Takayanagi conjecture relates the geodesic length to the entanglement entropy as shown in eq.5. This enables us to express the above eq. (33) which describes our holographic conjecture for the entanglement negativity in terms of the holographic mutual information between the pair of intervals  $(A, B_1)$  and  $(A, B_2)$  as follows

$$\mathcal{E} = \lim_{B \rightarrow A^c} \left[ \frac{3}{4} (\mathcal{I}(A, B_1) + \mathcal{I}(A, B_2)) \right], \quad (34)$$

$$\mathcal{I}(A, B_1) = S_A + S_{B_1} - S_{A \cup B_1} = \frac{1}{4G_N^3} (\mathcal{L}_A + \mathcal{L}_{B_1} - \mathcal{L}_{A \cup B_1}), \quad (35)$$

$$\mathcal{I}(A, B_2) = S_A + S_{B_2} - S_{A \cup B_2} = \frac{1}{4G_N^3} (\mathcal{L}_A + \mathcal{L}_{B_2} - \mathcal{L}_{A \cup B_2}). \quad (36)$$

Choosing the corresponding subsystems as shown in the fig.(1), the eq.(33) may now be used to compute the finite temperature entanglement negativity of the  $(1+1)$ -dimensional boundary CFT purely in terms of the bulk quantities. In the next section we will briefly review our results given in [22] where we have demonstrated that the above expression exactly matches with the large- $c$  limit of the entanglement negativity in  $CFT_{1+1}$  as given in [8].

In [22] we have shown that the above observations lead to a precise and elegant holographic conjecture for the entanglement negativity of higher dimensional ( $d > 2$ ) boundary CFTs at finite temperatures in a generic  $AdS_{d+1}/CFT_d$  scenario. To understand this it is required to partition a  $d$ -dimensional boundary CFT into the two subsystems  $A$  and its complement denoted as  $A^c$ . Subsequently we consider two other subsystems  $B_1$  and  $B_2$  adjacent to  $A$  and on either side of it such that  $B = (B_1 \cup B_2)$ . If  $\mathcal{A}_A$ ,  $\mathcal{A}_{B_1}$  and  $\mathcal{A}_{B_2}$  denote the areas of the co-dimension two static minimal surfaces in  $AdS_{d+1}$  anchored to the subsystems  $A$ ,  $B_1$  and  $B_2$  respectively, then the holographic entanglement negativity for the subsystem  $A$  is given by the following expression

$$\begin{aligned} \mathcal{E} &= \lim_{B \rightarrow A^c} \left[ \frac{3}{16G_N^{d+1}} (2\mathcal{A}_A + \mathcal{A}_{B_1} + \mathcal{A}_{B_2} - \mathcal{A}_{A \cup B_1} - \mathcal{A}_{A \cup B_2}) \right] \\ &= \lim_{B \rightarrow A^c} \left[ \frac{3}{4} (\mathcal{I}(A, B_1) + \mathcal{I}(A, B_2)) \right], \end{aligned} \quad (37)$$

where,  $G_N^{d+1}$  is the  $(d+1)$ -dimensional Newton constant and in the limit  $(B \rightarrow A^c)$  in eq.(37) corresponds to extending the subsystems  $B_1$  and  $B_2$  along one direction such that  $B = (B_1 \cup B_2)$  becomes infinitely large and in this limit it reduces to the complement  $A^c$ . In a later section, using the above mentioned holographic conjecture we will obtain the entanglement negativity for a  $d$ -dimensional boundary CFT at a finite temperature which is dual to a Schwarzschild black hole in  $AdS_{d+1}$  bulk space time. It will be demonstrated that the entanglement negativity at finite temperatures exhibits certain universal features independent of the dimensionality. This actually constitutes our main result presented in this article.

#### 4.1 Entanglement Negativity in $AdS_3/CFT_2$

According to the  $AdS/CFT$  correspondence, a  $(1+1)$ -dimensional CFT at a finite temperature ( $T = \beta^{-1}$ ) is dual to Euclidean BTZ black hole at the same Hawking temperature<sup>3</sup>. The metric for this Euclidean BTZ black hole is given by

$$ds^2 = (r^2 - r_h^2)d\tau_E^2 + \frac{R^2}{(r^2 - r_h^2)}dr^2 + r^2d\phi^2, \quad (38)$$

here,  $\tau_E$  is the compactified Euclidean time ( $\tau_E \sim \tau_E + \frac{2\pi R}{r_h}$ ) and  $\phi$  is a periodic coordinate. Under the co-ordinate transformation  $r = r_+ \cosh \rho$ ,  $\tau_E = \frac{R}{r_+} \theta$ ,  $\phi = \frac{R}{r_+} t$  the metric in (38) reduces to that of the Euclidean  $AdS_3$  as follows

$$ds^2 = R^2(d\rho^2 + \cosh^2 \rho dt^2 + \sinh^2 \rho d\theta^2). \quad (39)$$

The lengths of the geodesics in this geometry which are anchored to specific points on the  $AdS_3$  boundary is well known in these Euclidean Poincare co-ordinates [14]. From the above the length of the geodesic  $\mathcal{L}_\gamma$  anchored on the interval  $\gamma$  may be given as

$$\mathcal{L}_\gamma = 2R \ln \left[ \frac{\beta}{\pi a} \sinh \left[ \frac{\pi l_\gamma}{\beta} \right] \right], \quad (40)$$

here  $a$  is the UV cut-off for the boundary  $CFT_{1+1}$ ,  $R$  is the  $AdS_3$  length scale and  $l_\gamma$  represents the length of the subsystem- $\gamma$ . In the  $AdS_3/CFT_2$  scenario as shown in fig.(1) the geodesic length  $\mathcal{L}_\gamma$  given by eq.(40) may be identified for the intervals  $\gamma = \{A, B_1, B_2, A \cup B_1, A \cup B_2\}$ . Using the expression of the geodesic length given by eq.(40) and substituting it in eq.(33), the holographic entanglement negativity for the  $(1+1)$ -dimensional boundary CFT may be obtained as follows

$$\mathcal{E} = \lim_{L \rightarrow \infty} \left[ \frac{3R}{4G} \ln \left[ \frac{\beta}{\pi a} \frac{\sinh \left[ \frac{\pi(L - \frac{\ell}{2})}{\beta} \right] \sinh \left[ \frac{\pi \ell}{\beta} \right]}{\sinh \left[ \frac{\pi(L + \frac{\ell}{2})}{\beta} \right]} \right] \right]. \quad (41)$$

The above eq.(41) in the limit ( $L \rightarrow \infty$ ) reduces to

$$\mathcal{E} = \frac{c}{2} \ln \left[ \frac{\beta}{\pi a} \sinh \left[ \frac{\pi \ell}{\beta} \right] \right] - \frac{\pi c \ell}{2\beta}, \quad (42)$$

where we have made use of the previously mentioned Brown-Henneaux formula. Remarkably eq.(42) obtained from the bulk computation using our conjecture, matches exactly with the large- $c$  limit of the finite temperature result for the entanglement negativity of a  $(1+1)$ -dimensional CFT given by eq.(26). The above expression for the holographic entanglement negativity may be concisely expressed as

$$\mathcal{E} = \frac{3}{2} \left[ S_A - S_A^{th} \right]. \quad (43)$$

Here,  $S_A$  is the entanglement entropy and  $S_A^{th}$  is the thermal entropy of the subsystem  $A$ . The above expression conclusively demonstrates that the holographic entanglement negativity obtained from our conjecture captures the distillable quantum entanglement for the  $CFT_{1+1}$  at a finite temperature on the  $AdS_3$  boundary. Having reviewed the application of our holographic conjecture and its consequences in the  $AdS_3/CFT_2$  example we now proceed to the generic  $AdS_{d+1}/CFT_d$  example involving bulk  $AdS$  - Schwarzschild black holes. This is described in the next section.

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<sup>3</sup>The Euclidean BTZ black hole metric in the  $AdS_{2+1}$  bulk may be obtained by going from real time ( $\tau$ ) to imaginary time ( $\tau_E$ ), i.e.  $\tau \rightarrow i\tau_E$  such that the boundary  $CFT_{1+1}$  is now defined on the complex plane.

## 5 Holographic entanglement Negativity in $AdS_{d+1}/CFT_d$

In this section we apply our holographic conjecture to a generic  $AdS_{d+1}/CFT_d$  scenario to arrive at the entanglement negativity of a  $d$ -dimensional boundary  $CFT$  at a finite temperature dual to bulk  $AdS_{d+1}$  Schwarzschild black hole. In this case the  $d$ -dimensional boundary  $CFT$  is partitioned into the subsystem  $A$  and its complement  $A^c$ . In analogy with the  $AdS_3/CFT_2$  case we again consider two large but finite subsystems  $B_1$  and  $B_2$  adjacent to the subsystem  $A$  and on either side of it, such that  $B = (B_1 \cup B_2)$  as shown schematically in the fig.(2). The metric for a  $AdS_{d+1}$ -Schwarzschild black hole with a planar horizon in the Poincare coordinates is given by

$$ds^2 = -r^2\left(1 - \frac{r_h^d}{r^d}\right)dt^2 + \frac{dr^2}{r^2\left(1 - \frac{r_h^d}{r^d}\right)} + r^2 d\vec{x}^2, \quad (44)$$

where  $r_h$  is the horizon radius of the black hole with the Hawking temperature  $T = r_h d/4\pi$  and  $\vec{x} \equiv (x, x^i)$  are the spatial co-ordinates on the boundary and  $i = 1..(d-2)$ . Here we set the AdS length scale  $R$  to unity. The holographic entanglement negativity in this case is given by the eq.(37) in terms of the areas of the co dimension two extremal surfaces anchored on the corresponding subsystems and extending into the  $AdS_{d+1}$  bulk geometry (see fig.(2)) as follows

$$\mathcal{E} = \lim_{B \rightarrow A^c} \frac{3}{16G_N^{d+1}} \left[ 2\mathcal{A}_A + \mathcal{A}_{B_1} + \mathcal{A}_{B_2} - \mathcal{A}_{A \cup B_1} - \mathcal{A}_{A \cup B_2} \right], \quad (45)$$

$$\mathcal{E} = \lim_{B \rightarrow A^c} \frac{3}{4} \left[ \mathcal{I}(A, B_1) + \mathcal{I}(A, B_2) \right]. \quad (46)$$

Here,  $\mathcal{A}_\gamma$  is the area of the extremal surface anchored to the boundary of the corresponding subsystem ( $\gamma$ ) where  $\gamma = \{A, B_1, B_2, A \cup B_1, A \cup B_2\}$ . Note that as mentioned earlier,  $\mathcal{I}(A, B_1)$  and  $\mathcal{I}(A, B_2)$  are the holographic mutual information between the subsystems  $(A, B_1)$  and  $(A, B_2)$  respectively. As is evident from fig.(2) the subsystem  $A$  corresponds to a spatial region on the  $d$ -dimensional boundary defined by the coordinates  $x \in [-\frac{\ell}{2}, \frac{\ell}{2}]$ ,  $x^i \in [-\frac{L_2}{2}, \frac{L_2}{2}]$  where  $L_2$  is considered to be large. Similarly, the spatial region describing the subsystems  $B_1$  and  $B_2$  are defined by the coordinates  $x \in [-L, -\frac{\ell}{2}]$ ,  $x^i \in [-\frac{L_2}{2}, \frac{L_2}{2}]$  and  $x \in [\frac{\ell}{2}, L]$ ,  $x^i \in [-\frac{L_2}{2}, \frac{L_2}{2}]$  respectively. Note that from the above the spatial region corresponding to the subsystem  $A \cup B_1$  is defined by the coordinates  $x \in [-L, \frac{\ell}{2}]$ ,  $x^i \in [-\frac{L_2}{2}, \frac{L_2}{2}]$ .

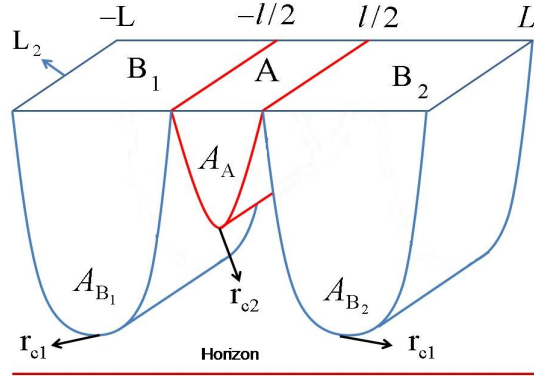


Figure 2: Schematic of extremal surfaces anchored on the subsystems  $A$ ,  $B_1$  and  $B_2$  which live on the  $(d)$ -dimensional boundary in the low temperature regime.

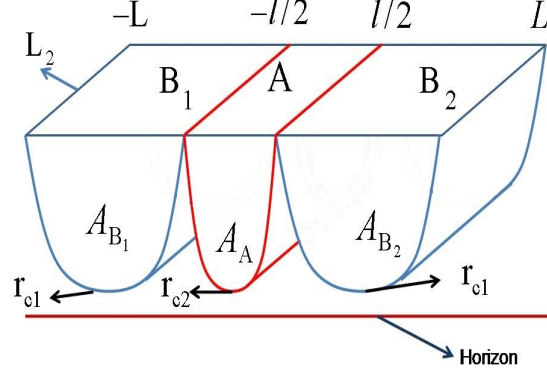


Figure 3: Schematic of extremal surfaces anchored on the subsystems  $A$ ,  $B_1$  and  $B_2$  which live on the  $(d)$ -dimensional boundary in the high temperature regime.

Notice that the subsystem  $A$  has been chosen to be symmetric along the partitioning direction as shown in the fig.(2). This leads to the equality of the extremal areas in eq.(46),  $\mathcal{A}_{B_1} = \mathcal{A}_{B_2}$  and  $\mathcal{A}_{A \cup B_1} = \mathcal{A}_{A \cup B_2}$ . This identification reduces the expression given in eq.(46) to the following form

$$\mathcal{E} = \lim_{B \rightarrow A^c} \frac{3}{8G_N^{d+1}} \left[ \mathcal{A}_A + \mathcal{A}_{B_1} - \mathcal{A}_{A \cup B_1} \right]. \quad (47)$$

The expression for the area of the surface which is anchored to a subsystem of the boundary  $d$ -dimensional  $CFT$  dual to a bulk planar  $AdS_{d+1}$ -Schwarzschild black hole is given in [17] as

$$\mathcal{A} = L_2^{d-2} \int dr r^{d-2} \sqrt{r^2 x'^2 + \frac{1}{r^2(1 - \frac{r_h^d}{r^d})}}. \quad (48)$$

Extremizing the above area integral leads to the following Euler-Lagrange equation

$$\int_{x_1}^{x_2} dx = \frac{2}{r_c} \int_0^1 \frac{u^{d-1} du}{\sqrt{(1 - u^{2d-2})}} \left(1 - \frac{r_h^d}{r_c^d} u^d\right)^{-\frac{1}{2}}, \quad (49)$$

here,  $x_1$  and  $x_2$  represent the end point of the subsystem under consideration,  $r_c$  represents the turning point of the extremal surface and the integration variable is given by  $u = \frac{r_c}{r}$ . After integration, the resulting equation may be inverted to obtain the turning radius  $r_c$ . This may then be substituted in the expression for the area of the extremal surface. The area integral in eq.(48) written in terms of the variable  $u$  may be expressed as

$$\mathcal{A} = 2L_2^{d-2} r_c^{d-2} \int_0^1 \frac{du}{u^{d-1} \sqrt{(1 - u^{2d-2})}} \left(1 - \frac{r_h^d}{r_c^d} u^d\right)^{-\frac{1}{2}}. \quad (50)$$

The integrals in eq.(50) and eq.(49) are not analytically solvable. Therefore to compute these integrals we adopt the method developed in [17] where the authors employ a certain expansion technique in terms of Gamma functions to compute these integrals order by order. Denoting the turning points of the extremal surfaces whose areas are given as  $\mathcal{A}_{B_1}$ ,  $\mathcal{A}_A$  and  $\mathcal{A}_{A \cup B_1}$  to be  $r_{c1}$ ,  $r_{c2}$  and  $r_{c3}$  respectively, it is possible to obtain the expression for the subsystem lengths using eq. (49) as follows [17]

$$L - \frac{\ell}{2} = \frac{2}{r_{c1}} \sum_{n=0}^{\infty} \left( \frac{1}{dn+1} \right) \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n+1)}{2(d-1)})}{\Gamma(\frac{(dn+1)}{2(d-1)})} \left( \frac{r_h}{r_{c1}} \right)^{nd}, \quad (51)$$

$$\ell = \frac{2}{r_{c2}} \sum_{n=0}^{\infty} \left( \frac{1}{dn+1} \right) \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n+1)}{2(d-1)})}{\Gamma(\frac{(dn+1)}{2(d-1)})} \left( \frac{r_h}{r_{c2}} \right)^{nd}, \quad (52)$$

$$L + \frac{\ell}{2} = \frac{2}{r_{c3}} \sum_{n=0}^{\infty} \left( \frac{1}{dn+1} \right) \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n+1)}{2(d-1)})}{\Gamma(\frac{(dn+1)}{2(d-1)})} \left( \frac{r_h}{r_{c3}} \right)^{nd}. \quad (53)$$

Whereas the expressions for the extremal surfaces  $\mathcal{A}_{B_1}$ ,  $\mathcal{A}_A$  and  $\mathcal{A}_{A \cup B_1}$  may be expressed as

$$\mathcal{A}_{B_1} = \frac{2}{d-2} \left( \frac{L_2}{a} \right)^{d-2} + 2L_2^{d-2} r_{c1}^{d-2} \sum_{n=0}^{\infty} \left( \frac{1}{2(d-1)} \right) \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n-1)+2}{2(d-1)})}{\Gamma(\frac{(dn+1)}{2(d-1)})} \left( \frac{r_h}{r_{c1}} \right)^{nd}, \quad (54)$$

$$\mathcal{A}_A = \frac{2}{d-2} \left( \frac{L_2}{a} \right)^{d-2} + 2L_2^{d-2} r_{c2}^{d-2} \sum_{n=0}^{\infty} \left( \frac{1}{2(d-1)} \right) \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n-1)+2}{2(d-1)})}{\Gamma(\frac{(dn+1)}{2(d-1)})} \left( \frac{r_h}{r_{c2}} \right)^{nd}, \quad (55)$$

$$\mathcal{A}_{A \cup B_1} = \frac{2}{d-2} \left( \frac{L_2}{a} \right)^{d-2} + 2L_2^{d-2} r_{c3}^{d-2} \sum_{n=0}^{\infty} \left( \frac{1}{2(d-1)} \right) \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n-1)+2}{2(d-1)})}{\Gamma(\frac{(dn+1)}{2(d-1)})} \left( \frac{r_h}{r_{c3}} \right)^{nd}. \quad (56)$$

It is to be noted that the integral for the area in eq.(50) is divergent and has to be regulated by an infrared cut-off of the bulk (say  $r_{in}$ ) which is related to the UV cut-off ( $a$ ) of the  $d$ -dimensional boundary CFT as  $r_{in} = 1/a$  [17]. Having performed all the integrals we substitute eq.(54), eq.(56) and eq.(55) in eq.(47) to arrive at the expression for entanglement negativity of the  $d$ -dimensional boundary CFT as

$$\begin{aligned} \mathcal{E} = \lim_{L \rightarrow \infty} \frac{3}{8G_N^{d+1}} & \left[ \frac{2}{d-2} \left( \frac{L_2}{a} \right)^{d-2} + 2L_2^{d-2} r_{c1}^{d-2} \sum_{n=0}^{\infty} \left( \frac{1}{2(d-1)} \right) \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n-1)+2}{2(d-1)})}{\Gamma(\frac{(dn+1)}{2(d-1)})} \left( \frac{r_h}{r_{c1}} \right)^{nd} \right. \\ & + 2L_2^{d-2} r_{c2}^{d-2} \sum_{n=0}^{\infty} \left( \frac{1}{2(d-1)} \right) \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n-1)+2}{2(d-1)})}{\Gamma(\frac{(dn+1)}{2(d-1)})} \left( \frac{r_h}{r_{c2}} \right)^{nd} \\ & \left. - 2L_2^{d-2} r_{c3}^{d-2} \sum_{n=0}^{\infty} \left( \frac{1}{2(d-1)} \right) \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n-1)+2}{2(d-1)})}{\Gamma(\frac{(dn+1)}{2(d-1)})} \left( \frac{r_h}{r_{c3}} \right)^{nd} \right]. \end{aligned} \quad (57)$$

Notice that it is required to invert the expressions in eq.(51), eq.(52) and eq.(53) to obtain  $r_{c1}$ ,  $r_{c2}$ ,  $r_{c3}$  and then substitute those in the above equation to obtain the holographic negativity as a function of the temperature and the length ( $l$ ) of the subsystem  $A$ .

## 5.1 Low temperature regime

In this section, we compute the holographic entanglement negativity of the  $d$ -dimensional boundary CFT in the low temperature regime. This regime corresponds to the temperature  $T\ell \ll 1$  which in the bulk translates to the case where the horizon is at a large distance from the turning point  $r_{c2}$  of the extremal surface anchored on the subsystem  $A$ . This is equivalent to the condition  $r_{c2} \gg r_h$  as shown in the fig.(2). As  $r_h \ell \ll 1$ , the expression for the turning point  $r_{c2}$  may be obtained perturbatively employing the technique described in [17] as follows

$$r_{c2} = \frac{b_0}{\ell} \left[ 1 + b_1 (r_h \ell)^d + O[r_h^{2d} \ell^{2d}] \right], \quad (58)$$

where  $b_0$ ,  $b_1$  are constants given by

$$b_0 = \frac{2\sqrt{\pi} \Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}, \quad (59)$$

$$b_1 = \frac{1}{2(d+1)} \frac{2^{\frac{1}{d-1}-d} \Gamma(1 + \frac{1}{2(d-1)}) \Gamma(\frac{1}{2(d-1)})^{d+1}}{\pi^{\frac{d+1}{2}} \Gamma(\frac{1}{2} + \frac{1}{(d-1)}) \Gamma(\frac{d}{2(d-1)})^d}. \quad (60)$$

We find the area  $\mathcal{A}_A$  by substituting the expression for  $r_{c2}$  given by eq.(58) in the eq.(55) while keeping only the leading and the sub leading terms in  $(r_h \ell)^d$  as follows

$$\mathcal{A}_A = \frac{2}{d-2} \left(\frac{L_2}{a}\right)^{d-2} + s_0 \left(\frac{L_2}{\ell}\right)^{d-2} \left[1 + s_1 (r_h \ell)^d + O[(r_h \ell)^{2d}]\right], \quad (61)$$

where  $s_0$  and  $s_1$  are given by

$$s_0 = \frac{2^{d-2} \pi^{\frac{d-1}{2}} \Gamma(-\frac{d-2}{2(d-1)}) \Gamma(\frac{d}{2(d-1)})^{d-2}}{(d-1) \Gamma(\frac{1}{2(d-1)})^{d-1}}, \quad (62)$$

$$s_1 = \frac{\Gamma(\frac{1}{2(d-1)})^{d+1}}{2^{d+1} \pi^{\frac{d}{2}} \Gamma(\frac{d}{2(d-1)})^d \Gamma(\frac{d+1}{2(d-1)})} \left( \frac{\Gamma(\frac{1}{d-1})}{\Gamma(-\frac{d-2}{2(d-1)})} + \frac{2^{\frac{1}{d-1}} (d-2) \Gamma(1 + \frac{1}{2(d-1)})}{\sqrt{\pi} (d+1)} \right). \quad (63)$$

The subsystems  $B_1$  and  $A \cup B_1$  in the  $d$ -dimensional boundary CFT with lengths  $(L - \ell/2)$  and  $L + \ell/2$  along the  $x$  direction are very large in the limit  $B \rightarrow A^c$  ( $L \rightarrow \infty$ ). Therefore, the extremal surfaces described by the areas  $\mathcal{A}_{B_1}$  and  $\mathcal{A}_{A \cup B_1}$  will extend deep into the bulk approaching the black hole horizon even at low temperatures i.e.,  $(r_{c1} \sim r_h)$  and  $(r_{c3} \sim r_h)$ . Hence in order to compute the expressions for the areas  $\mathcal{A}_{B_1}$  and  $\mathcal{A}_{A \cup B_1}$  we employ the method developed by the authors in [17] for the case when extremal surfaces approach the black hole horizon as described earlier. Through this procedure in our case we obtain the expression for the turning point  $r_{c1}$  for the extremal surface anchored on the subsystem  $B_1$  as follows

$$r_{c1} = r_h (1 + \epsilon_1), \quad (64)$$

$$\epsilon_1 = k_2 e^{-\sqrt{\frac{d(d-1)}{2}} r_h (L - \frac{\ell}{2})}, \quad (65)$$

where  $k_2$  is a constant given by

$$k_2 = \frac{1}{d} e^{\sqrt{\frac{d(d-1)}{2}} c_1}, \quad (66)$$

$$c_1 = \frac{2\sqrt{\pi} \Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{(d-1)})} + \sum_{n=1}^{\infty} \left( \frac{2}{(1+nd)} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n+1)}{2(d-1)})}{\Gamma(\frac{dn+1}{2(d-1)})} - \frac{\sqrt{2}}{\sqrt{d(d-1)}n} \right).$$

Substituting the expressions given by eq.(64) and eq.(65) in eq.(54) we obtain the area  $\mathcal{A}_{B_1}$  as an expansion in  $\epsilon_1$  up to  $O[\epsilon_1]$  as

$$\mathcal{A}_{B_1} = \frac{2}{d-2} \left(\frac{L_2}{a}\right)^{d-2} + \left[ L_2^{d-2} r_h^{d-1} (L - \frac{\ell}{2}) + L_2^{d-2} r_h^{d-2} (k_1 - \sqrt{\frac{2(d-1)}{d}} \epsilon_1) + O[\epsilon_1^2] \right], \quad (67)$$

where,  $k_1$  is a constant defined as

$$k_1 = 2 \left[ -\frac{\sqrt{\pi} (d-1) \Gamma(\frac{d}{2(d-1)})}{(d-2) \Gamma(\frac{1}{2(d-1)})} + \sum_{n=1}^{\infty} \frac{1}{1+nd} \left( \frac{d-1}{d(n-1)+2} \right) \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{d(n+1)}{2(d-1)})}{\Gamma(\frac{dn+1}{2(d-1)})} \right]. \quad (68)$$

Repeating the above procedure we find the expressions for  $r_{c3}$  and  $\mathcal{A}_{A \cup B_1}$  from eq.(53) and eq.(56) as follows

$$r_{c3} = r_h (1 + \epsilon_3), \quad (69)$$



$$\epsilon_3 = k_2 e^{-\sqrt{\frac{d(d-1)}{2}} r_h (L + \frac{\ell}{2})}, \quad (70)$$

$$\mathcal{A}_{A \cup B_1} = \frac{2}{d-2} \left( \frac{L_2}{a} \right)^{d-2} + \left[ L_2^{d-2} r_h^{d-1} \left( L + \frac{\ell}{2} \right) + L_2^{d-2} r_h^{d-2} \left( k_1 - \sqrt{\frac{2(d-1)}{d}} \epsilon_3 \right) + O[\epsilon_3^2] \right]. \quad (71)$$

Now we substitute the expressions given by eq.(67), eq.(61) and eq.(71) for the extremal areas  $\mathcal{A}_{B_1}$ ,  $\mathcal{A}_A$  and  $\mathcal{A}_{A \cup B_1}$  obtained in the low temperature regime, in eq.(47). This leads to the following expression for the entanglement negativity  $\mathcal{E}$  for the  $d$ -dimensional boundary CFT in the low temperature regime.

$$\mathcal{E} = \frac{3}{8G_N} \left[ \frac{2}{d-2} \left( \frac{L_2}{a} \right)^{d-2} + s_0 \left( \frac{L_2}{\ell} \right)^{d-2} [1 + s_1 (r_h \ell)^d] - V r_h^{d-1} \right], \quad (72)$$

where  $V = \ell L_2^{d-2}$  is the  $d$ -dimensional volume of the subsystem- $A$ . The above expression for the holographic entanglement negativity in the low temperature regime may be re expressed in a concise form as

$$\varepsilon = \frac{3}{2} \left[ S_A - S_A^{th} \right]. \quad (73)$$

In the above expression  $S_A$  is the entanglement entropy for the subsystem  $A$  in the  $d$ -dimensional boundary CFT dual to a  $AdS_{d+1}$ -Schwarzschild black hole and  $S_A^{th} = \frac{V r_h^{d-1}}{4G_N^{(d+1)}}$  represents the thermal entropy of the subsystem- $A$ . Remarkably, from the above equation it may be observed that as earlier for the  $AdS_3/CFT_2$  scenario, in this case also the entanglement negativity captures the distillable quantum entanglement through the removal of the thermal contribution in this regime. This confirms our holographic conjecture for the low temperature regime in the  $AdS_{d+1}/CFT_d$  scenario. We now extend our analysis to the high temperature regime in the next subsection.

## 5.2 High temperature regime

At high temperatures, the turning point  $r_{c2}$  of the extremal surface with the area  $\mathcal{A}_A$  approaches close to the black hole horizon which is described by the condition  $r_{c2} \sim r_h$  as shown in fig.(3). Note that the high temperature regime also implies a large horizon radius ( $r_h$ ) for the bulk  $AdS_{d+1}$ -Schwarzschild black hole. Following [17] we obtain  $\mathcal{A}_A$  in a near horizon expansion in  $\epsilon_2$  up to  $O[\epsilon_2]$  by considering  $r_{c2} = r_h(1 + \epsilon_2)$  as follows

$$r_{c2} = r_h(1 + \epsilon_2), \quad (74)$$

$$\epsilon_2 = k_2 e^{-\sqrt{\frac{d(d-1)}{2}} r_h \ell}, \quad (75)$$

$$\mathcal{A}_A = \frac{2}{d-2} \left( \frac{L_2}{a} \right)^{d-2} + \left[ L_2^{d-2} r_h^{d-1} (\ell) + L_2^{d-2} r_h^{d-2} \left( k_1 - \sqrt{\frac{2(d-1)}{d}} \epsilon_2 \right) + O[\epsilon_2^2] \right]. \quad (76)$$

We now turn to the evaluation of the other two extremal surfaces described by the areas  $\mathcal{A}_{B_1}$  and  $\mathcal{A}_{A \cup B_1}$ . Note that as described earlier these surfaces always probe the near horizon regime both at low and at high temperatures due to the limit  $B \rightarrow A^c$  or equivalently  $L \rightarrow \infty$ . Hence we may use the general expression for these extremal areas given in eq.(67) and eq.(71) in the high temperature regime as well. Following this we substitute the areas of all the three extremal surfaces given by eq.(76), eq.(67) and eq.(71) in the expression for the entanglement negativity given by eq.(47). This leads us to the expression for the entanglement negativity in the high temperature regime as follows

$$\mathcal{E} = \frac{3}{8G_N} \left[ \frac{2}{d-2} \left( \frac{L_2}{a} \right)^{d-2} + L_2^{d-2} r_h^{d-2} \left( k_1 - \sqrt{\frac{2(d-1)}{d}} k_2 e^{-\sqrt{\frac{d(d-1)}{2}} r_h (\ell)} \right) \right]. \quad (77)$$

Observe that as earlier for the low temperature regime we may re express the above equation in the high temperature regime also in the following concise form

$$\mathcal{E} = \frac{3}{2} \left[ S_A - S_A^{th} \right]. \quad (78)$$

From the above expression notice that as earlier for the low temperature regime, the entanglement negativity for the high temperature regime also leads to the distillable quantum entanglement through the removal of the thermal contribution. This indicates that the holographic entanglement negativity for the subsystem  $A$  in a generic  $AdS_{d+1}/CFT_d$  scenario involving bulk  $AdS_{d+1}$ -Schwarzschild black holes is given by the expression in eq.(78) for all temperatures confirming our conjecture. Naturally the results of the last two sections provide extremely strong evidence for the universality of our conjecture and its relevance to  $d$ -dimensional CFTs in a generic  $AdS_{d+1}/CFT_d$  scenario.

## 6 Summary and Conclusions

To summarize, in this article we have confirmed the recent holographic conjecture proposed by us Chaturvedi, Malvimat and Sengupta (CMS) in [22] for the entanglement negativity of  $d$ -dimensional conformal field theories at finite temperatures dual to bulk  $AdS_{d+1}$ -Schwarzschild black holes in a generic  $AdS_{d+1}/CFT_d$  scenario. Our conjecture relates the entanglement negativity to a certain algebraic sum of the areas of co dimension two extremal surfaces in the bulk geometry anchored on different subsystems in the boundary CFT which is equivalent to the sum of the holographic mutual information for the corresponding subsystems. In [22] we had shown in the context of the  $AdS_3/CFT_2$  scenario that the entanglement negativity computed through our prescription precisely captures the distillable quantum entanglement at all temperatures by the subtraction of the thermal contribution. Remarkably we have demonstrated in the context of our holographic conjecture that this feature is also valid for the entanglement negativity of  $d$ -dimensional CFTs at finite temperatures dual to  $AdS_{d+1}$ -Schwarzschild black holes. Hence this provides extremely strong evidence for the universality of our holographic conjecture for  $d$ -dimensional CFTs at finite temperatures. Naturally our conjecture for the entanglement negativity has significant implications in diverse fields such as quantum information theory, condensed matter physics and the information loss paradox for black holes. This significance arises from the fact that the entanglement negativity for the  $d$ -dimensional boundary CFTs at finite temperatures dual to  $AdS_{d+1}$ -Schwarzschild black holes computed through our conjecture clearly captures the distillable quantum entanglement involving the removal of the thermal contribution.

As mentioned earlier the issue of the purification leading to the distillable quantum entanglement is of central importance in the subject of quantum information theory recently. Our results presented in this article lead to a clear elucidation of this significant issue and constitutes a crucial and important contribution in the discipline which has witnessed intense research focus in the recent past. Furthermore it is also well known that the entanglement negativity is related to the topological order in diverse condensed matter systems and leads to the topological entropy for certain cases. Evidently our precise holographic prescription for this quantity would be of significant import in the investigation of such critical issues in condensed matter physics in the context of the  $AdS$  condensed matter theory (  $AdS/CMT$  ) correspondence. In fact it may be of crucial significance in the effort to obtain a theory of high temperature superconductivity, quantum quenches and thermalization which involve *entanglement evolution* and applications to the study of quantum phase transitions. It also well known that entanglement entropy and mutual information have played a important role in the investigation of the information loss paradox and the associated black hole *firewall problem*. Interestingly, our conjecture directly relates the entanglement negativity and the associated distillable quantum entanglement with the holographic mutual information. Naturally, this indicates that our proposed conjecture should also have sig-

nificant implications for the Information Loss Paradox and the black hole *firewall problem*. We hope to return to these interesting issues in the near future.

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## Appendix A Review of entanglement negativity in $CFT_{1+1}$

In this appendix, we review the procedure for obtaining the entanglement negativity in a  $(1+1)$ -dimensional CFT described by the authors Calabrese et al. in [8]. As discussed in the introduction, the entanglement negativity between two disjoint subsystems  $A_1$  and  $A_2$  ( $A = A_1 \cup A_2$ ) of a tripartite system where the rest of the system corresponds to  $A^c$  may be given as

$$\mathcal{E} = \log (Tr | \rho_A^{T_2} |), \quad (79)$$

here,  $\rho_A = Tr_{A^c}(\rho)$  is reduced density matrix and the operation of the partial transpose on this reduced density matrix  $\rho_A^{T_2}$  is described by eq.(2).

Note that for extended quantum many body systems like quantum field theories just as for entanglement entropy the computation of the entanglement negativity involves an infinite dimensional density matrix. Hence the application of the above formula for the entanglement negativity becomes problematic. However for  $(1+1)$  dimensional conformal field theories this issue may be addressed in the framework of the *replica technique* proposed in [8] mentioned earlier. Using this technique the authors were able to compute the entanglement negativity for a  $CFT_{1+1}$  by relating it to the quantity  $Tr(\rho_A^{T_2})^n$ . Their procedure for obtaining the entanglement negativity is based on the equality between  $Tr(\rho_A)^n$  and the following four point function of the twist and the anti-twist fields

$$Tr(\rho_A)^n = \langle \mathcal{T}_n(u_1) \bar{\mathcal{T}}_n(v_1) \mathcal{T}_n(u_2) \bar{\mathcal{T}}_n(v_2) \rangle. \quad (80)$$

In this regard, the operation of the partial transpose ( $\rho_A^{T_2}$ ) of the reduced density matrix  $\rho_A$  has the effect of exchanging upper and lower edges of the branch cut along the interval  $A_2$  on a  $n_e$ -sheeted Riemann surface. Thus the quantity  $Tr(\rho_A^{T_2})^n$  may be expressed in terms of the four point function of the twist and the anti-twist fields as

$$Tr(\rho_A^{T_2})^n = \langle \mathcal{T}_n(u_1) \bar{\mathcal{T}}_n(v_1) \bar{\mathcal{T}}_n(u_2) \mathcal{T}_n(v_2) \rangle. \quad (81)$$

It is to be noted that  $Tr(\rho_A^{T_2})^n$  shows different functional dependence on  $|\lambda_i|$  ( $\lambda_i$ 's are the eigenvalues of  $\rho_A^{T_2}$ ) depending on parity of  $n$ . However, the required expression for the entanglement negativity may be obtained in terms of  $Tr(\rho_A^{T_2})^n$  only if  $n = n_e(\text{even})$  [8]. Thus, by making use of the replica technique given in eq.(81), the authors defined the entanglement negativity between two disjoint intervals in a  $CFT_{1+1}$  as

$$\mathcal{E} = \lim_{n_e \rightarrow 1} \ln(Tr[(\rho_A^{T_2})^{n_e}]) = \lim_{n_e \rightarrow 1} \ln [\langle \mathcal{T}_{n_e}(u_1) \bar{\mathcal{T}}_{n_e}(v_1) \bar{\mathcal{T}}_{n_e}(u_2) \mathcal{T}_{n_e}(v_2) \rangle]. \quad (82)$$

### A.1 Entanglement negativity of a bipartite system at zero temperature

Here we explain the systematic method developed by the authors in [6, 7] in order to obtain the entanglement negativity in a  $CFT_{1+1}$  when the extended quantum system under consideration is bipartite and at zero temperature. In order to reduce a tripartite system  $(A_1, A_2, A^c)$  to a

bipartite configuration  $(A, A^c, \emptyset)$  where  $\emptyset$  is the null set, the authors make the identification  $u_2 \rightarrow v_1$  and  $v_2 \rightarrow u_1$  in eq.(82) such that the interval corresponding to the subsystem  $A$  is now a single interval denoted by  $[u, v]$ . With this identification, the correct form for the entanglement negativity of the subsystem  $A$  is given in terms of the two point function of the twist and the anti-twist fields as

$$\mathcal{E} = \lim_{n_e \rightarrow 1} \ln [Tr(\rho^{T_A})^{n_e}] = \lim_{n_e \rightarrow 1} \ln [\langle \mathcal{T}_{n_e}^2(u) \bar{\mathcal{T}}_{n_e}^2(v) \rangle], \quad (83)$$

where,  $\rho = \rho_{A \cup A^c}$  corresponds to the density matrix of the full system. In order to compute the two point function given in the equation above, the authors in [8] use the fact that the operator  $\mathcal{T}_j^2$  connects the  $j$ -th sheet of the Riemann surface to the  $(j+2)$ -th sheet. When the parity of  $n$  is even i.e  $n = n_e$ , the  $n_e$ -sheeted Riemann surface dissociates into two  $n_e/2$  sheeted Riemann surfaces which simplifies the expression for the entanglement negativity in eq.(83) as follows

$$\mathcal{E} = \lim_{n_e \rightarrow 1} \ln [\langle (\mathcal{T}_{n_e/2}(u) \bar{\mathcal{T}}_{n_e/2}(v))^2 \rangle]. \quad (84)$$

here the scaling dimension- $\Delta_{n_e}^{(2)}$  of the operator  $\mathcal{T}_{n_e}^2$  is related to the scaling dimension- $(\Delta_{n_e})$  of the operator  $\mathcal{T}_{n_e}$  as

$$\begin{aligned} \Delta_{n_e}^{(2)} = 2\Delta_{n_e/2} &= \frac{c}{6} \left( \frac{n_e}{2} - \frac{2}{n_e} \right), \\ \Delta_{n_e} &= \frac{c}{12} \left( n_e - \frac{1}{n_e} \right). \end{aligned} \quad (85)$$

Since the form of the two point function in eq.(84) is fixed in a  $(1+1)$ -dimensional CFT, we obtain the following expression for the zero temperature entanglement negativity through a straightforward substitution as

$$\mathcal{E} = \frac{c}{2} \ln \left( \frac{\ell}{a} \right) + constant, \quad (86)$$

where,  $\ell = |u - v|$  is the length of the subsystem- $A$  and  $a$  is the UV cutoff for the  $(1+1)$ -dimensional conformal field theory. From the above discussion one may observe that at zero temperature the entanglement negativity is equal to the Rényi entropy of order-1/2 which is a well known result for bipartite systems [3, 7].

## A.2 Entanglement negativity of a bipartite system at finite temperature

In this section, we review the procedure for the computation of finite temperature entanglement negativity of an extended bipartite system in a  $(1+1)$  dimensional CFT developed by the authors in [8]. Note that the method for obtaining the finite temperature entanglement negativity is subtle and the authors in [6] obtained an incorrect result which they had corrected in [8]. The reason for this error may be associated with the fact that the decoupling of the  $n_e$  sheeted Riemann surface into two  $n_e/2$  sheeted Riemann surfaces leads to a simplified expression for the entanglement negativity given in eq.(84) which is suitable for the zero temperature  $CFT_{1+1}$ . For finite temperatures where the  $CFT_{1+1}$  is considered on an infinitely long cylinder this form is unsuitable to compute the entanglement negativity. The authors in [8] noted that unlike the zero temperature case the four point function does not factorize into the square of two-point function given in eq.(84). Thus the expression for the entanglement negativity of a  $(1+1)$ -dimensional CFT involves the computation of the following four point function

$$\mathcal{E} = \lim_{L \rightarrow \infty} \lim_{n_e \rightarrow 1} \ln [Tr(\rho^{T_A})^{n_e}] = \lim_{L \rightarrow \infty} \lim_{n_e \rightarrow 1} \ln [\langle \mathcal{T}_{n_e}(-L) \bar{\mathcal{T}}_{n_e}^2(-\ell) \mathcal{T}_{n_e}^2(0) \bar{\mathcal{T}}_{n_e}(L) \rangle_\beta]. \quad (87)$$

In the above equation the interval corresponding to subsystem- $A$  is given by  $[u, v] = [-\ell, 0]$  whereas,  $\mathcal{T}_{n_e}(-L)$  and  $\mathcal{T}_{n_e}(L)$  correspond to the twist fields located at the end points of the

subsystems denoted as  $B_1 = [-L, -\ell]$  and  $B_2 = [0, L]$  at some large distance  $L$  from the interval  $A$ . Moreover, if we denote  $B = B_1 \cup B_2$  then the limit  $L \rightarrow \infty$  in eq.(87) corresponds to  $B \rightarrow A^c$ . Here, it is also to be noted that in order to get the correct result from eq.(87), the limit  $(L \rightarrow \infty)$  should be applied only after taking the replica limit  $(n_e \rightarrow 1)$ . The subscript  $\beta$  indicates that at finite temperatures it is required to evaluate the four point function in eq.(87) on an infinitely long cylinder of circumference  $\beta = 1/T$ . This cylindrical geometry may be obtained from the 2-dimensional complex plane by the following conformal transformation

$$z \rightarrow \omega = \frac{\beta}{2\pi} \ln z, \quad (88)$$

where,  $z$  denotes the coordinates on the complex plane and  $\omega$  denotes the coordinates on the cylinder. Under the conformal transformation given by eq.(88), the required four-point function of a  $CFT_{1+1}$  on the infinite cylinder is related to the four point function on the complex plane as follows

$$\langle \mathcal{T}_{n_e}(w_1) \bar{\mathcal{T}}_{n_e}^2(w_2) \mathcal{T}_{n_e}^2(w_3) \bar{\mathcal{T}}_{n_e}(w_4) \rangle_\beta = \prod_j |z'(w_j)|^{\Delta_j} \langle \mathcal{T}_{n_e}(z_1) \bar{\mathcal{T}}_{n_e}^2(z_2) \mathcal{T}_{n_e}^2(z_3) \bar{\mathcal{T}}_{n_e}(z_4) \rangle_{\mathbb{C}}, \quad (89)$$

here  $z'(w_j) = \frac{dz}{dw}|_{z=w_j}$  and  $\Delta_j$  is the scaling dimension of operator at  $w_j$ . The four point function in a  $(1+1)$ - dimensional CFT on the complex plane may be expressed as follows

$$\langle \mathcal{T}_{n_e}(z_1) \bar{\mathcal{T}}_{n_e}^2(z_2) \mathcal{T}_{n_e}^2(z_3) \bar{\mathcal{T}}_{n_e}(z_4) \rangle_{\mathbb{C}} = \frac{1}{z_{14}^{2\Delta_{n_e}} z_{23}^{2\Delta_{n_e}^{(2)}}} \frac{\mathcal{G}_{n_e}(x)}{x^{\Delta_{n_e} + \Delta_{n_e}^{(2)}}}, \quad x \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}. \quad (90)$$

In the above equation the  $z_i$ 's correspond to arbitrary complex numbers such that  $z_{ij} = |z_i - z_j|$  with  $\langle . \rangle$  standing for the expectation value. From eq.(90) it may be observed that the four point function is only fixed up to an undetermined function  $\mathcal{G}_{n_e}(x)$  of the cross-ratio  $x$ . The cross ratio  $x$  of the four points has two limits  $x \rightarrow 0$  and  $x \rightarrow 1$ , which correspond to high and low temperature limits respectively [8]. The asymptotic behaviour of the four point function mentioned above at low and high temperatures may be obtained through the OPE of  $\mathcal{T}_{n_e}(u) \bar{\mathcal{T}}_{n_e}(v)$ ,  $\mathcal{T}_{n_e}^2(u) \bar{\mathcal{T}}_{n_e}^2(v)$  and  $\mathcal{T}_{n_e}(u) \bar{\mathcal{T}}_{n_e}^2(v)$ . For low temperatures one has  $x \rightarrow 1$  i.e  $z_3 \rightarrow z_2$ ,  $z_4 \rightarrow z_1$  which leads to the following form of the four point function of the twist and the anti-twist fields

$$\langle \mathcal{T}_{n_e}(z_1) \bar{\mathcal{T}}_{n_e}^2(z_2) \mathcal{T}_{n_e}^2(z_3) \bar{\mathcal{T}}_{n_e}(z_4) \rangle = \langle \mathcal{T}_{n_e}(z_1) \bar{\mathcal{T}}_{n_e}(z_4) \rangle \langle \mathcal{T}_{n_e}^2(z_2) \bar{\mathcal{T}}_{n_e}^2(z_3) \rangle + \dots, \quad (91)$$

whereas for high temperatures  $x \rightarrow 0$  i.e  $z_2 \rightarrow z_1$ ,  $z_4 \rightarrow z_3$  which results in the following form of the four point function

$$\langle \mathcal{T}_{n_e}(z_1) \bar{\mathcal{T}}_{n_e}^2(z_2) \mathcal{T}_{n_e}^2(z_3) \bar{\mathcal{T}}_{n_e}(z_4) \rangle = \frac{C_{n_e}^2 c_{n_e}}{(z_{12}z_{34})^{\Delta_{n_e}^{(2)}} z_{13}^{2\Delta_{n_e}}} + \dots, \quad (92)$$

here  $c_{n_e}$  and  $C_{n_e}$  are constants that appear as the coefficients of the leading term in the OPE of the two point functions  $\mathcal{T}_{n_e}(u) \bar{\mathcal{T}}_{n_e}(v)$  and  $\mathcal{T}_{n_e}(u) \bar{\mathcal{T}}_{n_e}^2(v)$  respectively. The high and low temperature behaviour of the four point function given in eq.(91) and eq.(92) leads to following suggestive form of the four point function

$$\langle \mathcal{T}_{n_e}(z_1) \bar{\mathcal{T}}_{n_e}^2(z_2) \mathcal{T}_{n_e}^2(z_3) \bar{\mathcal{T}}_{n_e}(z_4) \rangle_{\mathbb{C}} = \frac{c_{n_e} c_{n_e/2}^2}{z_{14}^{2\Delta_{n_e}} z_{23}^{2\Delta_{n_e}^{(2)}}} \frac{\mathcal{F}_{n_e}(x)}{x^{\Delta_{n_e}^{(2)}}}, \quad x \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad (93)$$

where,  $c_{n_e/2}^2$  is a constant. Following [8], one may also obtain the constraints on the arbitrary function  $\mathcal{F}_{n_e}(x)$  in the two limits  $x \rightarrow 1$  and  $x \rightarrow 0$  as follows

$$\mathcal{F}_{n_e}(1) = 1, \quad \mathcal{F}_{n_e}(0) = \frac{C_{n_e}^2}{c_{n_e/2}^2}. \quad (94)$$

Rewriting  $z_i$ 's in eq.(93) in terms of the required coordinates on the infinite cylinder i.e  $(z_1, z_2, z_3, z_4) \rightarrow (e^{-2\pi L/\beta}, e^{-2\pi\ell/\beta}, 1, e^{2\pi L/\beta})$  and then using the transformation given by eq.(89) one may obtain the required four point function that may be related to the correct form of the entanglement negativity given by eq.(87) for a  $CFT_{1+1}$  at a finite temperature. Thus the finite temperature result for the entanglement negativity of a  $(1+1)$ -dimensional CFT due to Calabrese et al. [8] may be written down as follows

$$\mathcal{E} = \frac{c}{2} \ln \left[ \frac{\beta}{\pi a} \sinh \left( \frac{\pi\ell}{\beta} \right) \right] - \frac{\pi c\ell}{2\beta} + f(e^{-2\pi\ell/\beta}) + \ln(c_{1/2}^2 c_1). \quad (95)$$

here the undetermined function  $f(x)$  in the replica limit ( $n_e \rightarrow 1$ ) is defined as

$$f(x) = \lim_{n_e \rightarrow 1} \ln[\mathcal{F}_{n_e}(x)], \quad \lim_{L \rightarrow \infty} x = e^{-2\pi\ell/\beta} \quad (96)$$

It may be observed from the above equation that the function  $\mathcal{F}_{n_e}(x)$  becomes unity at zero temperature (i.e  $x = 1$ ) due to the constraint given by eq.(94). This implies that one should fix  $f(1) = \lim_{n_e \rightarrow 1} \ln \mathcal{F}_{n_e}(1) = 0$  in order to obtain correct result at zero temperature. It is to be noted that the second term in the eq.(95) corresponds to the thermal entropy of the subsystem  $A$  in the  $(1+1)$ -dimensional CFT and is related to the classical correlations. Remarkably, the removal of the second term in eq.(95) which corresponds to the thermal entropy of the subsystem  $A$  suggests that the entanglement negativity forms the correct measure of the distillable quantum entanglement even at finite temperatures.

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